حلول الفرض الأول التدريب الأول 2012

EXERCICE 1 (7 points)

Solution. By applying the formula $\binom{a}{b} = \dfrac{a!}{b!(a-b)!}$ we obtain the equation

$$1 + \frac{(n-k)^2}{(k+1)^2} = \frac{(n-k)^2(n-k-1)^2}{(k+1)^2(k+2)^2}\binom{n}{k+2}^2.$$

Hence $(k+2)^2\left[(k+1)^2 + (n-k)^2\right] = (n-k)^2(n-k-1)^2\binom{n}{k+2}^2$, which implies that $(k+1)^2 + (n-k)^2$ is a perfect square.

Let $(k+1)^2 + (n-k)^2 = t^2$, where $t \in \mathbb{N}$. We have

$$(k+2)t = (n-k)(n-k-1)\binom{n}{k+2} \geq 2\binom{n}{k+2}.$$

Using that $k+2 \leq n$ and $t = \sqrt{(k+1)^2 + (n-k)^2} < n+1$ we conclude that $(k+2)t \leq n^2$.

Let $3 \leq k+2 \leq n-3$ (the left hand side of this inequality follows from the condition of the problem). If $n \geq 6$, we have

$$2\binom{n}{k+2} \geq 2\binom{n}{3} = \frac{n(n-1)(n-2)}{3} > n^2,$$

i.e. the equation has no solution in this case.

When $k+2 = n-2$ we obtain $t^2 = (n-3)^2 + 16$, hence $t = 5$, $n = 6$, $k = 2$. Direct computation shows that $n = 6$ and $k = 2$ is not a solution. If $k+2 = n-1$ we have $t^2 = (n-2)^2 + 9$, so $t = 5$, $n = 6$, $k = 3$ and as above we conclude that no solution exists. Therefore positive integers n and k satisfying the equation do not exist.

Remark. The equation $x^2 + y^2 = z^4$ has infinitely many solutions in positive integers.

EXERCICE 2 (7 points)

Solution: Let $x^n f(y) - y^n f(x) = f\left(\frac{x}{y}\right)$.   (1)

Let substitute into (1) $x = y = 1$, then $f(1) = 0$.

Let substitute into (1) $y = 1$, then $f\left(\frac{1}{x}\right) = -f(x)$ (2).

Let substitute into (1) $y = 1/x$, then taking into account (2) we get $(x^2) = \left(x^n + \frac{1}{x^n}\right)f(x)$.   (3)

Let's change x by x^2 and y by y^2 in (1): $x^{2n}f(y^2) - y^{2n}f(x^2) = f\left(\frac{x^2}{y^2}\right)$ or (in view of (3)): $x^{2n}\left(y^n + \frac{1}{y^n}\right)f(y) - y^{2n}\left(x^n + \frac{1}{x^n}\right)f(x) = \left(\frac{x^n}{y^n} + \frac{y^n}{x^n}\right)f\left(\frac{y}{x}\right)$. (4)

By subtracting from (4) equation (1), multiplied by $\left(\frac{x^n}{y^n} + \frac{y^n}{x^n}\right)$, we can get: $y^n(1 - x^{2n})f(y) - x^n(1 - y^{2n})f(x) = 0$. Separating variables: $\frac{f(x)}{\frac{1}{x^n} - x^n} = \frac{f(y)}{\frac{1}{y^n} - y^n}$. Left side of this equality does not depend on y, and right side does not depend on x, therefore $\frac{f(x)}{\frac{1}{x^n} - x^n} = c$, where $\in \mathbb{R}$.

By simple check we can see that the function $f(x) = c(\frac{1}{x^n} - x^n)$ is indeed the solution of the equation.

Exercice 3 (7 points)

3. Let $r = 1/x, s = 1/y, t = 1/z$. There exists $\alpha < 1$ such that $r + s + t = \alpha^2 rst$ or $\alpha(r + s + t) = \alpha^3 rst$. Let $a = \alpha r$, $b = \alpha s$, $c = \alpha t$. Write $a = \tan A$, $b = \tan B$, $c = \tan C$, then $A + B + C = \pi$. It is clear that

$$\begin{aligned}\frac{1}{2} \times \text{LHS} &= \frac{1}{\sqrt{1+r^2}} + \frac{1}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+t^2}} \\ &< \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \\ &= \cos A + \cos B + \cos B \\ &\le 3\cos\left(\frac{A+B+C}{3}\right) = \frac{3}{2} = \frac{1}{2} \times \text{RHS}.\end{aligned}$$

2nd soln: Note that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz} \quad \Rightarrow \quad xy + yz + xz < 1.$$

Hence

$$\frac{2x}{\sqrt{1+x^2}} < \frac{2x}{\sqrt{x^2 + xy + xz + yz}} = \frac{2x}{\sqrt{(x+y)(x+z)}}.$$

By AM-GM we have

$$\frac{2x}{\sqrt{(x+y)(x+z)}} \le \frac{x}{x+y} + \frac{x}{x+z}.$$

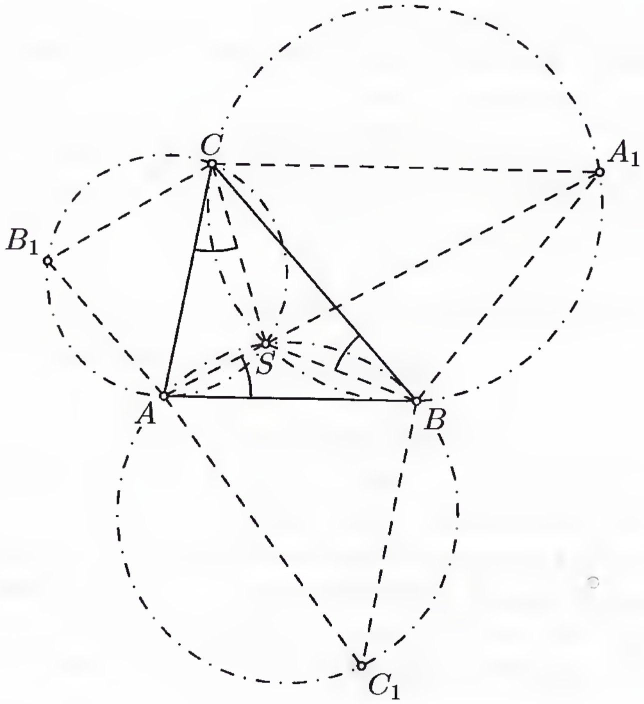
Similarly,

$$\frac{2y}{\sqrt{(y+z)(y+x)}} \le \frac{y}{y+z} + \frac{y}{y+x}, \quad \frac{2z}{\sqrt{(z+x)(z+y)}} \le \frac{z}{z+x} + \frac{z}{z+y}.$$

The desired inequality then follows by adding up the three inequalities.

Exercice 4 (7 points)

G3. Let us denote $\sphericalangle SAB = \sphericalangle SBC = \sphericalangle SCA = \varphi$, and the angles of the given triangle with α, β, γ.



Since $\sphericalangle CSA_1$ is the exterior angle of the triangle ACS we have

$$\sphericalangle CSA_1 = \sphericalangle CAS + \sphericalangle SCA = \sphericalangle CAS + \varphi = \sphericalangle CAS + \sphericalangle SAB = \sphericalangle CAB = \alpha.$$

Analogously, $\sphericalangle ASB_1 = \beta$ i $\sphericalangle BSC_1 = \gamma$.

Hence $\sphericalangle CBA_1 = \sphericalangle CSA_1 = \alpha$ and $\sphericalangle BCA_1 = \sphericalangle BSA_1 = \sphericalangle B_1SA = \beta$ so $\triangle BCA_1 \sim \triangle ABC$. Analogously, $\triangle B_1CA \sim \triangle ABC$ and $\triangle BC_1A \sim \triangle ABC$.

Therefore we have

$$\frac{P(BCA_1)}{P(ABC)} = \left(\frac{|BC|}{|AB|}\right)^2 = \frac{a^2}{c^2}$$

and analogously $\dfrac{P(B_1CA)}{P(ABC)} = \dfrac{b^2}{a^2}$ and $\dfrac{P(BC_1A)}{P(ABC)} = \dfrac{c^2}{b^2}$.

Finally,

$$\begin{aligned} P(A_1CB) + P(B_1AC) + P(C_1BA) &= \left(\frac{a^2}{c^2} + \frac{b^2}{a^2} + \frac{c^2}{b^2}\right) P(ABC) \\ &\geqslant 3\sqrt[3]{\frac{a^2}{c^2} \cdot \frac{b^2}{a^2} \cdot \frac{c^2}{b^2}}\, P(ABC) = 3P(ABC). \end{aligned}$$